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1980 J. Phys. A: Math. Gen. 13 2881

(http://iopscience.iop.org/0305-4470/13/9/015)

J. Phys. A: Math. Gen. 13 (1980) 2881–2888. Printed in Great Britain

Coefficients of fractional parentage for the states of an arbitrary number of $j = 1$ bosons

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Received 9 July 1979, in final form 14 April 1980

Abstract. The construction of symmetric states of N particles with angular momentum $j = 1$ is presented.

Elliott's result, dealing with the angular momentum content of the symmetric subset of states belonging to the set of states of N particles with angular momentum $j = 1$, is obtained using Racah algebra for the angular momentum coupling. The explicit functional dependence, on the total angular momentum, of the coefficients of fractional parentage (CFP) associated with one, two, three and four removed particles is found for arbitrary values of N. This dependence is expressed through the simple algebraic functions of J and N.

1. Introduction

One of the problems in group theory and its physical applications concerns the decomposition of the Kronecker product of N identical irreducible representations $D^{(j)}$ of the group $O^+(3)$ into irreducible representations (IR) of $SU(2j+1)$ and IR of $O^+(3)$. The particular case of this general problem consists in finding the multiplicities of IR of $O^+(3)$ in the IR of $SU(2j+1)$ (shortly, the angular momentum content of $SU(2j+1)$ irreducible representations). For solving this problem, Jahn (1950) suggested the recurrence method based on decomposition of the outer multiplications of IR of the permutation group S_N. An alternative approach, a simple graphical method, has recently been proposed by Mikhailov (1978). Recurrence relations obtained using combinatorial analysis, for the multiplicities of angular momenta in symmetric and antisymmetric IR of $SU(2j+1)$, are given by Buthner (1967).

The angular momentum content of the symmetric IR of SU(3) has been found by Elliott (1958a,b), with the aid of Littlewood's rule, for any value of $N^{a,b}$.

In this paper, we derive the recursion relations for the coefficients of fractional parentage (CFP) in symmetric states of N spin $j = 1$ particles. Those states belong also to the IR of $O^+(3)$. In this derivation we exploit the Racah algebra for the angular momentum coupling. The angular momentum content of the symmetric IR of SU(3) follows directly from the condition of the existence of the solutions of the homogeneous set of equations for the CFP coefficients (§ 2).

Using certain properties of symmetric states, we found the explicit functional dependence of the CFP on J and N, for any N and for $0 \leqslant J \leqslant N$ (i.e. we found the general solution of the recursion relations for CFP). We have also evaluated CFP for two, three and four particles in symmetric states.

0305-4470/80/092881+08\$01.50

2. The construction of symmetric states for N spin $j = 1$ particles

As usual, let us denote by $|j_i m_i\rangle$ the eigenstates of the operators $\hat{j}_i^2$ and $\hat{j}_i^z$ associated with the ith particle. In order to study the symmetric IR of the SU$(2j+1)$ and O$^+(3)$ groups, we will construct the Racah basis in the symmetric subspace of the Kronecker product representation $D^{(1)} \times D^{(1)} \times \ldots \times D^{(1)}$. Here the basic states are the eigenstates of the operators

$$\hat{J}^2 = \left(\sum_{i=1}^{N} \hat{j}_i\right)^2, \qquad \hat{J}^z = \left(\sum_{i=1}^{N} \hat{j}_i^z\right)^2. \tag{2.1}$$

For $N = 2$ the symmetric IR of SU$(2j+1)$ and O$^+(3)$ contain the eigenstates of $\hat{J}^2$ and $\hat{J}^z$ in which $J = 2 - 2k$ $(k = 0, 1)$. This result follows directly from the properties of Clebsch–Gordan coefficients $(j_1 j_2 m_1 m_2|JM)$ under the alternation of j_1 and j_2.

For $N > 2$, each symmetric state of N angular momenta, $|(j^N)f_N; JM\rangle$, can be written as a linear combination of products of symmetric states of $(N-1)$ angular momenta, $|(j^{N-1})f_{N-1}; J_{N-1}, M - m'\rangle$, by the state $|j_N m'\rangle$ of the Nth angular momentum:

$$\begin{aligned} |(j^N)f_N; JM\rangle &= \sum_{J_{N-1}} ((j^{N-1})J_{N-1}, j; J\|(j^N)J) \sum_{m'} |(j^{N-1})f_{N-1}; J_{N-1}, M - m'\rangle \\ &\quad \times |j_N m'\rangle (J_{N-1}, j_N, M - m', m'|JM) \\ &= \sum_{J_{N-1}} ((j^{N-1})J_{N-1}, j; J\|(j^N)J) |(j^{N-1})f_{N-1}; J_{N-1}, j_N; JM\rangle. \end{aligned} \tag{2.2}$$

The coefficients $((j^{N-1})J_{N-1}, j; J\|(j^N)J)$ are analogous to the similar coefficients in the expansion of the antisymmetric functions (Racah 1942a,b,c, 1951, Kaplan 1975) and are called the coefficients of fractional parentage (CFP).

We will introduce the shortened notation

$$f_N(J_{N-1}, J) = ((j^{N-1})J_{N-1}, j; J\|(j^N)J). \tag{2.3}$$

The recursion relation for the CFP coefficients is obtainable by substituting into (2.2) the expansion of the symmetric states $|(j^{N-1})f_{N-1}, J_{N-1}, j_N; JM\rangle$ through the CFP $f_{N-1}(J_{N-2}, J_{N-1})$, and by stating that the series obtained are independent of the choice of the removed angular momenta. This means that the states have the following property:

$$\begin{aligned} &\sum_{J_{N-1}} f_N(J_{N-1}, J) \sum_{J_{N-2}} f_{N-1}(J_{N-2}, J_{N-1}) |((j^{N-2})J_{N-2}, j_N)J_{N-1}, j_{N-1}; JM\rangle \\ &\quad = \sum_{J'_{N-1}} f_N(J'_{N-1}, J) \sum_{J_{N-2}} f_{N-1}(J_{N-2}, J'_{N-1}) |((j^{N-2})J_{N-2}, j_{N-1})J'_{N-1}, j_N; JM\rangle. \end{aligned} \tag{2.4}$$

By performing the Racah recoupling transformation

$$\begin{aligned} &|((j^{N-2})J_{N-2}, j_{N-1})J'_{N-1}, j_N; JM\rangle \\ &\quad = (-1)^{J + J'_{N-1} - 2j - J_{N-2}} \sum_{J_{N-1}} (-1)^{J_{N-1}} |((j^{N-2})J_{N-2}, j_N)J_{N-1}, j_{N-1}; JM\rangle \\ &\quad \times [(2J'_{N-1} + 1)(2J_{N-1} + 1)]^{1/2} W(jJ_{N-2}Jj; J'_{N-1}J_{N-1}) \end{aligned} \tag{2.5}$$

we obtain the following set of equations for the coefficients $f_N(J_{N-1}, J)$ and $f_{N-1}(J_{N-2}, J_{N-1})$:

$$\begin{aligned} f_N(J_{N-1}, J) f_{N-1}(J_{N-2}, J_{N-1}) &= (-1)^{J_{N-1}+J-J_{N-2}-2j} \sum_{J'_{N-1}} (-1)^{J'_{N-1}} f_N(J'_{N-1}, J) f_{N-1}(J_{N-2}, J'_{N-1}) \\ &\quad \times [(2J_{N-1}+1)(2J'_{N-1}+1)]^{1/2} W(jJ_{N-2}Jj; J'_{N-1}J_{N-1}). \end{aligned} \tag{2.6}$$

The indexes J_{N-1} and J_{N-2} in (2.6) take the values $(J-1, J, J+1)$ and $(J_{N-1}-1, J_{N-1}, J_{N-1}+1)$, respectively.

For $j=1$ and $N=3$, one easily finds the solutions of equations (2.6):

$$\begin{array}{lll} f_3(1, J)=0, & J=3,2,1,0, & \\ f_3(J_2, 2)=0, & f_3(J_2, 0)=0, & f_3(2,3)=1, \\ f_3(0,3)=0, & f_3(0,1)=\sqrt{5}/3, & f_3(2,1)=2/3. \end{array} \tag{2.7}$$

This result shows that the angular momentum content of the symmetric IR of SU(3) in the case $N=3$ is $J=1,3$.

With the aid of the results given in (2.7) we find that for $N=4$ the system of equations (2.6) has no solutions for $J=3,1$. For $J=4,2,0$ we find:

$$\begin{array}{llll} f_4(3,4)=1, & f_4(2,4)=0, & f_4(1,4)=0, & f_4(0,4)=0, \\ f_4(3,2)=\sqrt{3}/\sqrt{10}, & f_4(2,2)=0, & f_4(1,2)=\sqrt{7}/\sqrt{10}, & f_4(0,2)=0, \\ f_4(3,0)=0 & f_4(2,0)=0 & f_4(1,0)=1 & f_4(0,0)=0. \end{array} \tag{2.8}$$

These results suggest that the symmetric states of $M\,j=1$ angular momenta are those for which the total angular momentum, $J_M=J$, takes the values

$$J=M-2k, \qquad k=0,1,2,\ldots,[M/2]. \tag{2.9}$$

In order to prove this conjecture we will use the method of mathematical induction. Let us suppose that the conjecture and its consequences are true for $M=(N-1)$ angular momenta. Then we shall prove that from (2.6) it follows that this conjecture is true for $M=N$ angular momenta, too.

The supposition that (2.9) is valid for $N-1$ angular momenta implies the assumption that

$$\begin{aligned} &f_{N-1}(J_{N-2}, J_{N-1})=0 \qquad \text{if } J_{N-1}=N-1-(2k_1+1) \qquad \text{or } J_{N-2}=N-2-(2k_2+1), \\ &\qquad k_1=0,1,2,\ldots,\left[\frac{N-1}{2}\right], \qquad k_2=0,1,\ldots,\left[\frac{N-1}{2}\right]. \end{aligned} \tag{2.10}$$

Now, the set of equations (2.6) can be written explicitly

$$\begin{aligned} &f_N(J,J)=0, \\ &-f_N(J+1,J) f_{N-1}(J,J+1)[(2J+1)(2J+3)]^{1/2} W(1JJ1; J+1, J) \\ &\qquad = f_N(J-1,J) f_{N-1}(J,J-1)[(2J+1)(2J-1)]^{1/2} W(1JJ1; J-1, J), \\ &f_N(J-1,J) f_{N-1}(J,J-1)[1-(2J-1) W(1JJ1; J-1, J-1)] \\ &\qquad = f_N(J+1,J) f_{N-1}(J,J+1)[(2J-1)(2J+3)]^{1/2} W(1JJ1; J+1, J-1), \end{aligned} \tag{2.11}$$

$$f_N(J+1, J) f_{N-1}(J, J+1)[1-(2J+3)W(1JJ1; J+1, J+1)]$$
$$= f_N(J-1, J) f_{N-1}(J, J-1)[(2J+3)(2J-1)]^{1/2} W(1JJ1; J-1, J+1).$$

In the remaining equations each term equals zero, and therefore they are identically satisfied.

The set of equations (2.11) has non-trivial solutions if the following conditions are fulfilled:

$$\frac{1-(2J+3)W(J11J; J+1, J+1)}{[(2J+1)(2J+3)]^{1/2} W(1JJ1; J+1, J)} = -\frac{(2J+3)^{1/2}}{(2J+1)^{1/2}} \frac{W(1JJ1; J-1, J+1)}{W(1JJ1; J-1, J)}, \qquad (2.12a)$$

$$\frac{1-(2J-1)W(1JJ1; J-1, J-1)}{W(1JJ1; J-1, J)} = -(2J-1)\frac{W(1JJ1; J+1, J-1)}{W(1JJ1; J+1, J)}, \qquad (2.12b)$$

$$1-(2J-1)W(1, J-2, J, 1; J-1, J-1) = 0, \qquad (2.12c)$$

$$(2J+3)W(1, J+2, J, 1; J+1, J+1) = 1. \qquad (2.12d)$$

It is easy to check the validity of these conditions by writing the explicit algebraic expressions for Racah coefficients as functions of J (Brink and Satchler 1962). One notices that the relations (2.12a) and (2.12b) between Racah coefficients are not Racah–Elliott relations.

The conditions (2.12) being fulfilled, the coefficients f_N can be determined using any equation from the set (2.11). From the first and the second equations in the set (2.11) one finds:

$$\frac{f_N(J+1, J)}{f_N(J-1, J)} = \frac{(2J-1)^{1/2}}{(2J+3)^{1/2}} \frac{J+1}{J} \frac{f_{N-1}(J, J-1)}{f_{N-1}(J, J+1)},$$
$$f_N(J, J) = 0. \qquad (2.13)$$

For $J = N$ and $J = 0$ the coefficients f_N have the values

$$\begin{aligned} J &= N: \quad f_N(N+1, N) = 0, f_N(N-1, N) = 1, \\ J &= 0: \quad f_N(1, 0) = 1, f_N(0, 0) = 0, \text{ for } N \text{ even}, \\ J &= 0: \quad f_N(1, 0) = 0, f_N(0, 0) = 0, \text{ for } N \text{ odd}. \end{aligned} \qquad (2.14)$$

Combining (2.10) with (2.13) we find:

$$f_N(J_{N-1}, J_N) = 0 \qquad \text{if } J_N = N-(2k_1+1), J_{N-1} = N-1-(2k_2+1). \qquad (2.15)$$

In this way we have shown that if the conjecture (2.9) is true for $M = N-1$ angular momenta, it is true for $M = N$ angular momenta, too. This is the result obtained previously by Elliott (1958) with the aid of Littlewood's rule.

The relations (2.13) allow the determination of the coefficients $f_N(J_{N-1}, J)$ if the coefficients $f_{N-1}(J_{N-2}, J_{N-1})$ are known.

The dimension of the symmetric irreducible representation belonging to the Kronecker product of N irreducible representations $D^{(1)}$ equals $\binom{N+2}{N}$. The number of linearly independent symmetric states constructed above is

$$\sum_{k=0}^{[N/2]} [2(N-2k)+1] = \frac{N^2+3N+2}{2}.$$

This means that the constructed set of symmetric state covers the whole symmetric subspace.

3. The elevation of algebraic expressions for fractional parentage coefficients associated with one, two, three and four removed particles of angular momentum $j=1$

If N is very large, a lengthy procedure has to be accomplished in order to evaluate the coefficients $f_N(J_{N-1}, J)$ from the recursion relation (2.13). But, due to the fact that for a given J and N there are only two unknown coefficients $f_N(J-1, J)$, $f_N(J+1, J)$, it turns out that it is possible to determine algebraic expressions for those coefficients by using the following property of symmetric states:

$$\langle (j^N)f_N; JM|\hat{j}^z_N|(j^N)f_N; JM\rangle = \frac{1}{N-1}\langle (j^N)f_N; JM\Big|\sum_{i=1}^{N-1}\hat{j}^z_i|(j^N)f_N; JM\rangle. \tag{3.1}$$

By expressing the left- and right-hand sides of (3.1) through the coefficients $f_N(J-1, J)$ and $f_N(J+1, J)$ and using the algebraic expressions for the CG coefficients, one finds

$$M\left(\frac{f_N^2(J-1,J)}{J}-\frac{f_N^2(J+1,J)}{J+1}\right)=\frac{M}{N-1}\left(f_N^2(J+1,J)\frac{J+2}{J+1}+f_N^2(J-1,J)\frac{J-1}{J}\right) \tag{3.2}$$

and consequently

$$\frac{f_N^2(J+1,J)}{f_N^2(J-1,J)}=\frac{J+1}{J}\,\frac{N-J}{N+J+1}. \tag{3.3}$$

With the aid of the condition of normality one obtains

$$\begin{aligned} f_N(J-1,J) &\equiv ((j^{N-1})J-1, j; J\|(j^N)J)=\left(\frac{J}{2J+1}\,\frac{N+J+1}{N}\right)^{1/2},\\ f_N(J+1,J) &\equiv ((j^{N-1})J+1, j; J\|(j^N)J)=\left(\frac{J+1}{2J+1}\,\frac{N-J}{N}\right)^{1/2}. \end{aligned} \tag{3.4}$$

It is easy to see that the functions (3.4) satisfy the recursive relation (2.13).

Now, one can evaluate the coefficients $((j^{N-r})J_{N-r}(j^r)J_r; J\|(j^N)J)$, for any r in the series

$$|(j^N)f_N; JM\rangle=\sum_{J_{N-r},J_r}((j^{N-r})J_{N-r},(j^r)J_r; J\|(j^N)J)|(j^{N-r})J_{N-r}, f_{N-r}; (j^r)J_r, f_r; JM\rangle. \tag{3.5}$$

In order to accomplish this task for $r=2$, it is necessary to write twice the series (2.2). Then one finds

$$\begin{aligned} b_N(J_{N-2}, J_2, J) &\equiv ((j^{N-2})J_{N-2},(j^2)J_2; J\|(j^N)J)\\ &=\sum_{J_{N-1}} f_{N-1}(J_{N-2}, J_{N-1})f_N(J_{N-1}, J)[(2J_2+1)(2J_{N-1}+1)]^{1/2}\\ &\quad\times W(J_{N-2}jJj; J_{N-1}J_2). \end{aligned} \tag{3.6}$$

Similarly, for $r=3$ and $r=4$ one evaluates

$$
\begin{aligned}
d_N(J_{N-3}, J_3, J) & \\
&\equiv ((j^{N-3})J_{N-3}, (j^3)J_3; J \| (j^N)J) \\
&= \frac{1}{f_3(J_2, J_3)} \sum_{J_{N-2}} f_{N-2}(J_{N-3}, J_{N-2})[(2J_{N-2}+1)(2J_3+1)]^{1/2} \\
&\quad \times b_N(J_{N-2}, J_2, J) W(J_{N-3} j J J_2; J_{N-2} J_3), \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
e_N(J_{N-4}, J_4, J) & \\
&\equiv ((j^{N-4})J_{N-4}, (j^4)J_4; J \| (j^N)J) \\
&= \frac{1}{b_4(J'_2, J_2, J)} \sum_{J_{N-2}} b_{N-2}(J_{N-4}, J'_2, J_{N-2}) b_N(J_{N-2}, J_2, J) \\
&\quad \times W(J_{N-4}, J'_2 J J_2; J_{N-2} J_4)[(2J_4+1)(2J_{N-2}+1)]^{1/2}. \qquad (3.8)
\end{aligned}
$$

From (3.4), (3.6), (3.7) and (3.8) one finds the expressions for the coefficients $((j^{N-r})J_{N-r}, (j^r)J_r; J\|(j^N)J)$, $r=2, 3, 4$, listed in the appendix.

Acknowledgement

We wish to thank Dr Dj Šijački for useful information.

Appendix

$$
\begin{aligned}
b_N(J,0,J) &= \left(\frac{(N-J)(N+J+1)}{3N(N-1)}\right)^{1/2} \\
b_N(J-2,2,J) &= \left(\frac{(N+J+1)(N+J-1)}{N(N-1)}\right)^{1/2}\left(\frac{J(J-1)}{(2J+1)(2J-1)}\right)^{1/2} \\
b_N(J+2,2,J) &= \left(\frac{(N-J)(N-J-2)}{N(N-1)}\right)^{1/2}\left(\frac{(J+1)(J+2)}{(2J+1)(2J+3)}\right)^{1/2} \\
b_N(J,2,J) &= \left(\frac{2}{3}\right)^{1/2}\left(\frac{(N+J+1)(N-J)}{N(N-1)}\right)^{1/2}\left(\frac{J(J+1)}{(2J-1)(2J+3)}\right)^{1/2} \qquad (A.1)
\end{aligned}
$$

$$
\begin{aligned}
d_N(J+3,3,J) &= \left(\frac{(N-J)(N-J-2)(N-J-4)}{N(N-1)(N-2)}\right)^{1/2}\left(\frac{(J+1)(J+2)(J+3)}{(2J-1)(2J+1)(2J+3)}\right)^{1/2} \\
d_N(J+1,3,J) &= \left(\frac{3}{5}\right)^{1/2}\left(\frac{(N-1)(N-J-2)(N+J+1)}{N(N-1)(N-2)}\right)^{1/2}\left(\frac{J(J+1)(J+2)}{(2J-1)(2J+1)(2J+5)}\right)^{1/2} \\
d_N(J+1,1,J) &= \left(\frac{3}{5}\right)^{1/2}\left(\frac{(N-J)(N-J-2)(N+J+1)}{N(N-1)(N-2)}\right)^{1/2}\left(\frac{J+1}{2J+1}\right)^{1/2} \\
d_N(J-1,3,J) &= \left(\frac{3}{5}\right)^{1/2}\left(\frac{(N+J+1)(N+J-1)(N-J)}{N(N-1)(N-2)}\right)^{1/2}\left(\frac{J(J+1)(J-1)}{(2J+1)(2J-3)(2J+3)}\right)^{1/2}
\end{aligned}
$$

$$d_N(J-1,1,J)=\left(\frac{3}{5}\right)^{1/2}\left(\frac{(N-J)(N+J+1)(N+J-1)}{N(N-1)(N-2)}\right)^{1/2}\left(\frac{J}{2J+1}\right)^{1/2}$$

$$d_N(J-3,3,J)=\left(\frac{(N+J+1)(N+J-1)(N+J-3)}{N(N-1)(N-2)}\right)^{1/2}\left(\frac{J(J-1)(J-2)}{(2J-1)(2J+1)(2J-3)}\right)^{1/2} \tag{A.2}$$

$$\begin{aligned} e_N(J+4,4,J) &= \left(\frac{(N-J)(N-J-2)(N-J-4)(N-J-6)}{N(N-1)(N-2)(N-3)}\right)^{1/2} \\ &\quad\times\left(\frac{(J+1)(J+2)(J+3)(J+4)}{(2J+1)(2J+3)(2J+5)(2J+7)}\right)^{1/2} \end{aligned}$$

$$\begin{aligned} e_N(J+2,4,J) &= \frac{2}{\sqrt{7}}\left(\frac{(N-J)(N-J-2)(N+J-1)(N-J-4)}{N(N-1)(N-2)(N-3)}\right)^{1/2} \\ &\quad\times\left(\frac{J(J+1)(J+2)(J+3)}{(2J+1)(2J-1)(2J+7)(2J+3)}\right)^{1/2} \end{aligned}$$

$$\begin{aligned} e_N(J+2,2,J) &= \left(\frac{2\times 3}{7}\right)^{1/2}\left(\frac{(N-J)(N-J-2)(N-J-4)(N+J+1)}{N(N-1)(N-2)(N-3)}\right)^{1/2} \\ &\quad\times\left(\frac{(J+1)(J+2)}{(2J+1)(2J+3)}\right)^{1/2} \end{aligned}$$

$$\begin{aligned} e_N(J,4,J) &= \frac{3\sqrt{2}}{(5\times 7)^{1/2}}\left(\frac{(N-J)(N-J-2)(N+J+1)(N+J-1)}{N(N-1)(N-2)(N-3)}\right)^{1/2} \\ &\quad\times\left(\frac{J(J+1)(J-1)(J+2)}{(2J-1)(2J+3)(2J-3)(2J+5)}\right)^{1/2} \end{aligned}$$

$$e_N(J,2,J)=\frac{2}{\sqrt{7}}\left(\frac{(N-J)(N-J-2)(N+J+1)(N+J-1)}{N(N-1)(N-2)(N-3)}\right)^{1/2}\left(\frac{J(J+1)}{(2J-1)(2J+3)}\right)^{1/2}$$

$$e_N(J,0,J)=\frac{1}{\sqrt{5}}\left(\frac{(N-J)(N-J-2)(N+J+1)(N+J-1)}{N(N-1)(N-2)(N-3)}\right)^{1/2}$$

$$\begin{aligned} e_N(J-2,4,J) &= \frac{2}{\sqrt{7}}\left(\frac{(N-J)(N+J-1)(N+J-3)(N+J+1)}{N(N-1)(N-2)(N-3)}\right)^{1/2} \\ &\quad\times\left(\frac{J(J-1)(J+1)(J-2)}{(2J+1)(2J-1)(2J+3)(2J-5)}\right)^{1/2} \end{aligned}$$

$$e_N(J-2,2,J) = \frac{(2\times 3)^{1/2}}{\sqrt{7}}\left(\frac{(N-J)(N+J+1)(N+J-1)(N+J-3)}{N(N-1)(N-2)(N-3)}\right)^{1/2} \times\left(\frac{J(J-1)}{(2J+1)(2J-1)}\right)^{1/2}$$

$$e_N(J-4,4,J) = \left(\frac{(N+J+1)(N+J-1)(N+J-3)(N+J-5)}{N(N-1)(N-2)(N-3)}\right)^{1/2} \times\left(\frac{J(J-1)(J-2)(J-3)}{(2J-1)(2J+1)(2J-3)(2J-5)}\right)^{1/2}. \tag{A.3}$$

For all other values of J_{N-r} and J_r the coefficients $((J^{N-r})J_{N-r}, (j^r)J_r; J\|(j^N)J)$ are identically zero.

References

Brink D M and Satchler G R 1962 *Angular Momentum* (Oxford: Clarendon)
Buthner H 1967 *Z. Phys.* **198** 949–507
Elliott J P 1958a *Proc. R. Soc.* A **245** 128–54
—— 1958b *Proc. R. Soc.* A **245** 562–81
Hamermesh M 1962 *Group Theory and Its Applications to Physical Problems* (Reading, Mass: Addison-Wesley)
Jahn H A 1950 *Proc. R. Soc.* **201** 516–44
Kaplan I G 1975 *Symmetry of Many-Electrons Systems* (New York: Academic)
Mikhailov V V 1978 *J. Phys. A: Math. Gen.* **11** 443–56
Racah G 1942a *Phys. Rev.* **61** 134–45
—— 1942b *Phys. Rev.* **62** 146–70
—— 1942c *Phys. Rev.* **63** 171–200
—— 1951 *Lectures Notes on Group Theory and Spectroscopy* (Princeton: Institute for Advanced Study)